Exercise sheet 9: Black holes

Please prepare your solutions, ready to present in the class on **29.06.2022** at **16:00**.

1. You are in a rocket ship which has just crossed the event horizon of a Schwarzschild black hole. You have plenty of fuel remaining, and would like to live as long as possible (i.e. you want to maximise the time it takes to reach the singularity at $r = 0$).

   (a) What worldline should you take? Is this worldline a geodesic?
   (*Hint: use ordinary Schwarzschild coordinates* (t, r, θ, ϕ)*, and parameterise your worldline by the coordinate* r*.*)

   (b) If the mass of the black hole is 1 billion solar masses ($M = 10^9 M_\odot \approx 1.99 \times 10^{39}$ kg, the typical mass for black holes in galactic centres), how many hours do you have from crossing the event horizon to hitting the singularity?

2. Consider an observer sitting at constant spatial coordinates (r_o, θ_o, ϕ_o) around a Schwarzschild black hole of mass M. The observer drops a beacon into the black hole (straight down, along a radial trajectory) which emits radiation at a constant wavelength λ_e (in the beacon's rest frame).

   (a) Determine the coordinate speed $\frac{\mathrm{d}r}{\mathrm{d}t}$ of the beacon as a function of r.

   (b) Imagine there is another observer at fixed r, with a locally inertial coordinate system set up as the beacon passes by. Calculate the proper speed of the beacon, that is, the coordinate speed measured by this second observer, as a function of r. Evaluate this speed at the event horizon.

   (c) Calculate the wavelength λ_o, as measured by the observer at $r = r_o$, in terms of the the radius r at which the radiation was emitted.

   (d) Calculate the time t_o at which a beam emitted by the beacon at radius r will be observed by the observer at $r = r_o$.

   (e) Show that at late times, the redshift grows exponentially as $\frac{\lambda_o}{\lambda_e} \propto e^{t_o/T}$, for some time constant T which should be determined in terms of the black hole mass M.

3. Consider $\mathrm{d}s^2 = -\mathrm{d}t^2 + (1 - \lambda r^2)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$, where $\lambda > 0$ is constant.

   (a) Show that the null geodesics for this metric (in the plane $\theta = \frac{\pi}{2}$) satisfy

   $$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = r^2(1 - \lambda r^2)(\mu r^2 - 1), \quad \mu \text{ constant.}$$

   (b) Using part (a), determine the shape of the paths that light rays follow in this spacetime.